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Some Results and Problems on the Diophantine Equations

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1. On Diophantine equations $x_1^{x_1} x_2^{x_2} \dots x_k^{x_k} = z^z$

Erdős asked for integer solutions of the equation

$$x^x y^y = z^z \quad (1)$$

with $x > 1, y > 1$. In 1940, Ko Chao^[1] proved that when $(x, y) = 1$, equation (1) has no solutions in positive integers $x > 1, y > 1, z > 1$ and when $(x, y) \neq 1$, equation (1) has infinitely many solutions

$$\begin{aligned} x &= 2^{2^{n+1}(2^n - n - 1) + 2n} (2^n - 1)^{2(2^n - 1)} \\ y &= 2^{2^{n+1}(2^n - n - 1)} (2^n - 1)^{2(2^n - 1) + 2} \\ z &= 2^{2^{n+1}(2^n - n - 1) + n + 1} (2^n - 1)^{2(2^n - 1) + 1} \end{aligned}$$

$n > 1$, with $4xy = z$. Other solutions have not been found yet. Recently, Erdős pointed out that it is possible that these should be all the solutions of the equation (1).

In 1984, Uchiyama^{[(内山) [2]]} proved that there can only be a finite number of solutions for any fixed value of $Q = xy/z < \frac{1}{4}$.

Anderson conjectured that the equation $w^w x^x y^y = z^z$ has no solution with $1 < w < x < y$.

In 1964, Ko Chao and Sun Qi^[3] proved the equation

$$\prod_{i=1}^k x_i^{x_i} = z^z, \quad x_i > 1, k \geq 2, i = 1, \dots, k,$$

has infinitely many solutions

$$\begin{aligned}
x_1 &= k^{k^n(k^{n+1}-2n-k)+2n} (k^n-1)^{2(k^n-1)} \\
x_2 &= k^{k^n(k^{n+1}-2n-k)} (k^n-1)^{2(k^n-1)+2} \\
x_3 &= \dots = x_k = k^{k^n(k^{n+1}-2n-k)+n} (k^n-1)^{2(k^n-1)+1} \\
z &= k^{k^n(k^{n+1}-2n-k)+n+1} (k^n-1)^{2(k^n-1)+1},
\end{aligned}$$

of which the first one is $x_1=3^{14}2^4$, $x_2=3^{12}2^6$, $x_3=3^{13}2^5$, $z=3^{14}2^5$ for $k=3$. It gives a counter-example to Anderson's Conjecture.

For $2 \nmid xy$ are there any solutions of equation (1)? This still remains unproved. For the equation $x_1^{x_1} x_2^{x_2} x_3^{x_3} = z^z$ we asked that are there any solution with $1 < x_1 < x_2 < x_3$, $2 \nmid x, x_2, x_3$?

2. Some exponential equations

Jesmanowicz conjectured that the diophantine equation

$$a^x + b^y = c^z$$

has no integer solution except $x=y=z=2$, where a, b, c satisfy $a^2+b^2=c^2$. Ko Chao^{[4]-[6]} made a lot of investigations about it in 1958-1965. Lu Wenduan^[7], Chen Jingrun^[8] and Sun Qi^[9] also have studied this conjecture. For example, Lu proved that if $a=4n^2-1$, $b=4n$, $c=4n^2+1$, then Jesmanowicz's conjecture is true.

For the Diophantine equation

$$a^x + b^y = c^z \quad (2)$$

where a, b, c are different primes. In 1958-1976, Nagell, (石谷野) (内4) Makowski, Hadano, Uchiyama studied this equation. They gave all the solution (x,y,z) for $\max(a,b,c) \leq 17$. In 1984, Sun Qi

and Zhou Xiaoming^[10] gave all the nonnegative integral solutions of equation (2) for $\max(a,b,c)=19$. We proved also that the equation (2) has no solutions in non-negative integers $x>1, y, z$, if that $a=2, b=p, c=q$, where $p \equiv 5 \pmod{8}$ or $p \equiv 1 \pmod{8}$ and $p=u^2+16v^2, 2 \nmid v$, and $q \equiv 3 \pmod{4}, q \equiv 2 \pmod{p}$.

In 1985, for $\max(a,b,c)=23$,^{it} has been solved by Yang Xiaozuo.^[11]

In 1987, for $29 \leq \max(a,b,c) < 100$,^{it} has been solved by Cao Zhenfu.^[12]

Selfridge asks for what a and b

$$2^a - 2^b \mid n^a - n^b, \quad (3)$$

is true for all n ?

Sun Qi and Zhang Mingzhi^[13] proved that for $0 \leq b < a$ if and only if $(a,b)=(1,0), (2,1), (3,1), (4,2), (5,3), (5,1), (6,2), (7,3), (8,4), (8,2), (9,3), (14,2), (15,3), (16,4)$, (3) is true for all n .

3. Some diophantine equations which arise in the combinatorial theory and the theory of finite groups

Hall asked for the integer solution of the Diophantine equation

$$p^r + 2 = q^s \quad (4)$$

which arises in the combinatorial theory, where p, q are prime numbers. This includes that $5^2 + 2 = 3^3$, but we know no other case in which both $r>1$ and $s>1$. Sun Qi and Zhou Xiaoming^[14] studied the case $p+2=q$ in (4). Cao Zhenfu^[15] proved that when $p+2=q$, then the equation (4) has no solution for $r>1, s>1$.

Crescenzo^[16] investigated the equation (5) below, which arise in the theory of finite groups.

$$p^m - 2q^n = \pm 1, \quad p, q \text{ primes}, m > 1, n > 1. \quad (5)$$

Crescenzo proved that with the exception of the relation $(239)^2 - 2(13)^4 = -1$, every solution of (5) has exponents $m=n=2$.

However, it should be noted that Crescenzo's theorem is wrong. Because the equation (5) has another solution $p=3$, $q=11$, $m=5$, $n=2$.

For diophantine equation

$$3^m - 2q^n = 1, \quad m > 1, n > 1, q \text{ is an odd prime}, 2 \nmid m, \quad (6)$$

we conjectured that equation (6) has no solution except $q=11$, $m=5$, $n=2$.

For equation (6), the proof of following results are easy.

1) If $2 \mid n$, then the conjecture is true.

2) If the equation (6) has solution, then $q \equiv 1 \pmod{12}$.

3) If the equation (6) has solution then $\left(\frac{q}{7}\right) = \left(\frac{q}{13}\right) = \left(\frac{q}{757}\right) = 1$, where $\left(\frac{a}{q}\right)$ denotes legendre symbol.

Recently, Sun Qi studies further equations (7) and (8) below, which include (4) and (6) respectively.

For diophantine equation

$$a^m - kb^n = 2, \quad k > 0, 2 \nmid k, \quad (7)$$

and diophantine equation

$$a^m - lb^n = 1, \quad l > 0, \quad (8)$$

we proved the following results.

1) If the equation (7) has positive integer solution a_1, b_1, m_1, n_1 , with $2 \nmid m_1, n_1, m_1 > 1, n_1 > 1$, then $\frac{a_1^{m_1} + kb_1^{n_1}}{2} + a_1^{\frac{m_1-1}{2}} b_1^{\frac{n_1-1}{2}} \sqrt{ka_1 b_1}$ is the fundamental solution of the Pell's equation $x^2 - ka_1 b_1 y^2 = 1$.

2) If the equation (8) has positive integer solution a_2, b_2, m_2, n_2 , with $2 \nmid m_2 n_2, m_2 > 1, n_2 > 1$, then $a_2^{m_2} + \ell b_2^{n_2} + 2a_2^{\frac{m_2-1}{2}} b_2^{\frac{n_2-1}{2}}$ is the fundamental solution of the Pell's equation $x^2 - 1\ell a_2 b_2 y^2 = 1$.

From above results, we proved that equation (7) has no positive integer solution a_1, b_1, m_1, n_1 , $2 \nmid m_1 n_1, m_1 > 1, n_1 > 1$, if that $q_1 = kb_1 t^2 + 2$ or $b_1 = ka_1 t^2 + 2$, etc. For the equation (8), we proved similar results also. If $q = pt^2 + 2$, where p, q denote odd prime numbers, then equation $q^m = p^n + 2$ has no integer solution m, n with $m > 1, n > 1, 2 \nmid mn$.

We proved also the following result.

If $q = 6s^2 + 1$, then the equation (6) has no solution.

4. Some Cubic equations and Quartic equations

From the well known identity

$$(x+1)^3 + (x-1)^3 - 2x^3 = 6x,$$

Mordell suggested that perhaps most of the numbers can be expressed as $x^3 + y^3 + 2z^3$ with integers x, y, z . In 1936, Ko Chao^[18] gave the decompositions into four cubes in this form for $n \leq 100$ except the numbers 76, 99.

For the diophantine equations $x^3 + y^3 + 2z^3 = 76$ and $x^3 + y^3 + 2z^3 = 99$, we asked that are there any integral solution x, y, z ? This still remains unproved.

A interesting equation is

$$x^3 + y^3 + z^3 = n \tag{9}$$

When $n=3$, there are solutions given by $(x,y,z)=(1,1,1)$, $(4,4,-5)$, $(4,-5,4)$, $(-5,4,4)$. In 1984, Scarowsky and Boyarsky proved that the equation (9) has no new solutions were found for $|m| \leq 50000$, where $x+y+z=3m$, $m \in \mathbb{Z}$. In 1985, Cassels proved that any integral solution of the equation (9) has $x \equiv y \equiv z \pmod{9}$. Recently, Sun Qi^[19] proved that if $n=9a^3$, where a is not divisible by primes of the form $6k+1$, then any integral solution of the equation $x^3+y^3+z^3=9a^3$ satisfies $9 \mid \frac{xyz}{d^3}$, where $(x,y,z)=d$. If $n=3a^3$, $3 \nmid a$, then any integral solution of the equation $x^3+y^3+z^3=3a^3$ satisfies $\frac{x}{d} \equiv \frac{y}{d} \equiv \frac{z}{d} \pmod{9}$.

Ljunggren proved that if $D > 2$ be a square-free integer which is not divisible by primes of the form $6n+1$, then equations

$$x^3 \pm 1 = Dy^2 \quad (10)$$

have at most one solution in positive integers x, y . In 1981, Ko Chao and Sun Qi^{[20],[21]} proved that the only solution in integers of the equations (10) is $x=1, y=0$. In 1975-1981, Ko Chao and Sun Qi^{[22]-[26]} studied the equation

$$x^4 - Dy^2 = 1, \quad D > 1, \quad \mu(D) \neq 0. \quad (11)$$

We They proved that 1) If $D \equiv 3 \pmod{8}$, $\xi = x_0 + y_0 \sqrt{D}$ is the fundamental solution of the equation $x^2 - Dy^2 = 1$ and if $x_0 \equiv 0 \pmod{2}$, then the equation (11) has no solutions in positive integers x, y . 2) If $D=2p$, p is an odd prime number, then the equation (11) has no positive integral solutions except $p=3$, $x=7, y=20$. 3) If D is not divisible by primes of the form

$4n+1$, then the equation (11) has no positive integral solutions. In 1979 and 1981, Ko Chao and Sun Qi^{[27]-[29]} also studied the equations $x^2 - Dy^4 = 1$, $x^4 + 4 = Dy^2$ and $x^3 \pm 8 = Dy^2$.

For equation $6y^2 = x(x+1)(2x+1)$ Mordell asked if there was an elementary proof. In 1985, Ma Degang^[30] have answered the Mordell's question.

In 1942, Ljunggren ~~has~~ showed that the only solutions of $x^2 = 2y^4 - 1$ in positive integers are (1,1) and (239,13) but his proof is difficult. Mordell asks if it is possible to find a simple or elementary proof. This still remains ~~un~~proved.

In 1967, Bumby proved that the diophantine equation

$$2y^2 = 3x^4 - 1 \quad (12)$$

has only integer solutions $x = \pm 1, \pm 3$. The proof depends upon an application of the (law quadratic reciprocity) in the quadratic fields $\mathbb{Q}(\sqrt{-2})$.

In 1979, Bremner proved that the diophantine equation

$$3x^4 - 4y^4 - 2x^2 + 12y^2 - 9 = 0 \quad (13)$$

only positive integer solutions $x=1, y=1$, and $x=3, y=3$. The proof depends upon an application of the Skolem's p-adic method.

We asks if it is possible to find an elementary proof for the equation (12) or the equation (13).

5. The equation $\frac{1}{x_1} \dots \frac{1}{x_s} \pm \frac{1}{x_1 \dots x_s} = 1$. Znām problem.
S-problem.

For the equation

$$\sum_{i=1}^s \frac{1}{x_i} - \frac{1}{x_1 \dots x_s} = 1, \quad 0 < x_1 < \dots < x_s, \quad (14)$$

in 1964, Ko Chao and Sun Qi^[31] gave all solutions for $s=5$ and $s=6$.

Let $\Omega(s)$ be the number of positive integral solutions of the equation (14). In 1978, Sun Qi^[32] proved that when $s \geq 4$, then $\Omega(s) < \Omega(s+1)$.

In 1978, Janák and Shula gave eighteen solutions of the system of congruences

$$x_1 \dots x_{i-1} x_{i+1} \dots x_n + 1 \equiv 0 \pmod{x_i}, \quad x_i > 1, \quad i=1, \dots, n, \quad n > 1, \quad (15)$$

for $n=7$. Let $H(n)$ be the number of solutions of the system of congruences (15). In 1983, Sun Qi^[33] proved that if $n \geq 4$, then $H(n) < H(n+1)$. As a corollary one obtains: If $n \geq 7$, then $H(n) \geq n+11$.

In 1972, Znām asked whether for every positive integer $n > 1$ there exist integers $x_i > 1$ ($i=1, \dots, n$) such that x_i is a proper divisor of the numbers $x_1 \dots x_{i-1} x_{i+1} \dots x_n + 1$ for every i . In 1983, Sun Qi^[34] proved that let $Z(n)$ be the number of solutions of the Znām problem with $1 < x_1 < \dots < x_n$, we have $Z(n) \geq \Omega(n) - \Omega(n-1) > 0$, when $n \geq 5$. Hence the problem of Znām is completely solved. It is difficult to prove $Z(n+1) > Z(n)$, when $n \geq 5$.

In 1985, Sun Qi and Cao Zhenfu^[35] studied the equation

$$\sum_{i=1}^s \frac{1}{x_i} - \frac{1}{x_1 \dots x_s} = 1, \quad 0 < x_1 \dots < x_s. \quad (16)$$

Let $A(s)$ be the number of solutions of the equation (16).

We **They** proved that if $t \geq 9$, then $A(t+1) \geq \Omega(t) + \Omega(t-1) + 6$.

For $s=6$, we gave that there are 17 solutions of equation (16) in all. I conjecture that if $n \geq 3$, then $A(n+1) > A(n)$.

In 1984, Sun Qi and Cao Zhenfu^[36] proved that $\Omega(s+1) \geq \Omega(s)+3$, when $s \geq 10$. From this result, we also have $Z(s) \geq 3$ and $H(s) \geq 3s-9$, when $s \geq 10$.

In 1986, Sun Qi and Cao Zhenfu^[37] proved the following theorems.

1) If $s \geq 2$, then $Z(s) = H(s) - H(s-1)$.

2) If $s \geq 10$, then $\Omega(s+1) \geq \Omega(s)+5$.

3) If $s \geq 11$, then $Z(s) \geq 5$.

For $Z(s)$, we conjecture that if $s \geq 4$, then

$$Z(s+1) > Z(s).$$

In 1984, Sun Qi posed such a problem that, for each integer $n > 1$, if there are n integers $x_i > 1$ ($i=1, \dots, n$) such that each x_i is a proper ^{divisor} of integer $x_1 \dots x_{i-1} \wedge x_{i+1} \dots x_n - 1$. We call the problem as \mathfrak{S} -problem for simplicity. For each integer $n > 1$, we use the symbol $X(n)$ to express the number of solutions to \mathfrak{S} -problem for the case of n integers. Recently, Li-Shuguang^[38] proved that if $n \geq 4$, then $X(n) > 0$ and if $n=2, 3$, then $X(n)=0$. Thus, the \mathfrak{S} -problem is solved. Is this true that $X(n+1) > X(n)$ for $n \geq 4$?

6. The equation $\sum_{i=1}^n \frac{y_i}{d_i} \equiv 0 \pmod{1}$. Diophantine equation over Finite fields.

Let d_1, \dots, d_n be fixed positive integers. It is well-known that the number $I(d_1, \dots, d_n)$ of solutions of the equation

$$\frac{y_1}{d_1} + \frac{y_2}{d_2} + \dots + \frac{y_n}{d_n} \equiv 0 \pmod{1}, \quad y_i \text{ integers},$$

$$1 \leq y_i < d_i \quad (i=1, \dots, n) \quad (17)$$

play an important role in the study of diagonal equations over finite fields.

In 1948 and in 1949, Hua and Vandiver,^{[39],[40]} Furtado, Weil at about the same time proved the following results. If N denote the number of solutions of the equation

$$\sum_{i=1}^n a_i x_i^{d_i} = 0, \quad \text{where } d_i \mid q-1 \quad (i=1, \dots, n) \quad (18)$$

over finite field F , then

$$|N - q^{n-1}| \leq I(d_1, \dots, d_n) (q-1) q^{\frac{n-2}{2}} \quad (19)$$

Hence the value of $I(d_1, \dots, d_n)$ heavily affects the estimate of the number N of solutions of the equation (18).

For the case $d_1 = d_2 = \dots = d_n$, it is proved that $I(d, \dots, d) = \frac{d-1}{d} ((d-1)^{n-1} + (-1)^n)$. For the more general case $d_1 \mid d_2 \dots d_n$, in 1986, Sun Qi, Wan Daqing, Ma Degang^[41] proved that

$$I(d_1, \dots, d_n) = \prod_{j=1}^{n-1} (d_j - 1) - \prod_{j=1}^{n-2} (d_j - 1) \dots + (-1)^{n-1} (d_2 - 1)(d_1 - 1) +$$

$(-1)^n (d_1 - 1)$. A complicated formula for $I(d_1, \dots, d_n)$ was obtained independently by Lide and Niederreiter, Stanly, and us^[42] with different methods. The formula can be stated as follows

$$I(d_1, \dots, d_n) = (-1)^n + \sum_{r=1}^n (-1)^{n-r} \sum_{1 \leq i_1 < \dots < i_r \leq n} \frac{d_{i_1} \dots d_{i_r}}{\text{lcm}[d_{i_1}, \dots, d_{i_r}]}$$

Form (19), it is interesting to determine when $I(d_1, \dots, d_n) = 0$, for if $I(d_1, \dots, d_n) = 0$, then (18) has exactly q^{n-1} solutions. Some partial results have been obtained by Joly. In 1985, Sun Qi and Wan Daqing^[43] proved following theorem: Let $n > 2$, then (17) has no solutions if and only if one of the

following conditions holds. 1) For some $d_i, (d_i, \frac{d_1 \cdots d_n}{d_i}) = 1$ or 2) If $d_{i_1}, \dots, d_{i_k} (1 \leq i_1 < \dots < i_k \leq n)$ is the set of all even integers among $\{d_1, \dots, d_n\}$, then $2 \nmid k$, $\frac{d_{i_1}}{2}, \dots, \frac{d_{i_k}}{2}$ are pairwise prime, and d_{i_j} is prime to any odd number in $\{d_1, \dots, d_n\} (j=1, \dots, k)$ if $k < n$.

Recently, Sun Qi^[44] have proved the following theorems.

Theorem 1. Suppose $GF(p)$ is a finite field, where p is an odd prime number, $\{u, v, \theta\} \subset GF(p)$, $uv \neq 0$, if

$$p > \sqrt{p} (2^{\omega(p-1)} - 1)^2 + \sqrt{p} - (\sqrt{p} - 3) \frac{(p-1)}{\varphi(p-1)} - 1,$$

then there are two primitive roots α and β in $GF(p)$ such that $u\alpha + v\beta = \theta$, where μ is Mobius function, φ is Euler's totient function, $\omega(p-1)$ denotes the number of distinct prime factors of $p-1$.

Theorem 2. If $p > 2^{60}$, then there are two primitive roots α and β such that $u\alpha + v\beta = \theta$.

Theorem 3. If $p > 3$, and

$$p > \sqrt{p} (2^{\omega(p-1)} - 1)^2 - (\sqrt{p} - 1) \left(\frac{(p-1)}{\varphi(p-1)} - 1 \right),$$

then there are two primitive roots α and β such that $\alpha - \beta = 1$.

Vegh asks whether, for all primes $p > 61$, every integer can be expressed as the difference of two primitive roots of p .

From theorem 2, we easily deduce the following corollary.

Corollary. If $p > 2^{60}$, then every integer can be expressed as the difference of two primitive roots of p .

We can extend theorems 1-3 to $GF(p^n)$ ($n > 1$) without difficulty. For example, we have the following theorem.

Theorem 4. If $p^n > 2^{60}$, then there are two primitive roots

α and β in $GF(p^n)$ such that $u\alpha + v\beta = 0$, where $\{u, v, 0\} \subset GF(p^n)$, $uv \neq 0$.

In order to prove our theorems we need the following lemmas.

Lemma 1. Let χ and λ be characters of $GF(p)$, and set

$$J_{u,v}(\chi, \lambda, \theta) = \sum_{\substack{u\ell + vm = \theta \\ \ell, m \in GF(p)}} \chi(\ell) \lambda(m), \text{ then}$$

$$|J_{u,v}(\chi, \lambda, \theta)| = \begin{cases} \sqrt{p}, & \text{if } \chi\lambda \neq \chi_0. \\ 1, & \text{if } \chi\lambda = \chi_0. \end{cases}$$

where χ_0 denotes the trivial character of $GF(p)$.

Lemma 2. If $\delta > 1$, $\eta > 1$, $\delta | p-1$, $\eta | p-1$, $(a, \delta) = (b, \eta) = 1$, $1 \leq a \leq \delta$, $1 \leq b \leq \eta$, then $\chi_{a(p-1)/\delta}$, $\chi_{b(p-1)/\eta}$ are not equal to χ_0 , and $\chi_{a(p-1)/\delta} \chi_{b(p-1)/\eta} = \chi_0$ if $\frac{a}{\delta} + \frac{b}{\eta} \equiv 0 \pmod{1}$, $\chi_{a(p-1)/\delta} \chi_{b(p-1)/\eta} \neq \chi_0$, if $\frac{a}{\delta} + \frac{b}{\eta} \not\equiv 1 \pmod{1}$.

Lemma 3. Let $n \in GF(p)$, $n \neq 0$, then

$$\sum_{\substack{k|p-1 \\ (a,k)=1}} \frac{\mu(k)}{\phi(k)} \sum_{\substack{a=1 \\ (a,k)=1}}^k e^{\frac{2\pi i a \ln n}{k}} = \begin{cases} 0, & \text{if } n \text{ is not a primitive root,} \\ \frac{p-1}{\phi(p-1)}, & \text{if } n \text{ is a primitive root,} \end{cases}$$

Lemma 4.

$$\sum_{\delta|p-1} \frac{|\mu(\delta)|}{\phi(\delta)} = \frac{p-1}{\phi(p-1)}.$$

A natural problem is that, is the result above true to primitive roots modulo p^l ($l \geq 2$, the p is an odd prime)? The problem is solved in [45]. Sun Qi and Li Shuguang have proved the following theorem.

Theorem. Let p be an odd prime and integer $l \geq 2$. When $p > 2^{60}$, there exist at least $(p-2)p^{l-2}$ pairs of primitive roots

α and β modulo p^l such that

$$a\alpha + b\beta \equiv c \pmod{p^l}.$$

Recently, Sun Qi, ^[46] have proved the following theorem.

Theorem Let p be an odd prime and $h \in \text{GF}(p)$, $h \neq 0$.

If $m \geq 3$, then there is a primitive root g in $\text{GF}(p^m)$ such that

$$g + g^p + \dots + g^{p^{m-1}} = h$$

except $m=3$, $p=11$.

For $m=3$, $p=11$, the problem above still remains unproved.

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